Regularity properties on the real line

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Hejnice

Some weak forms of the Axiom of Choice:

- The **Weak Axiom of Choice wAC** says that for any countable family of non-empty subsets of a given set of power $2^{\aleph_0}$ there exists a choice function.

- The **Axiom of Dependent Choice DC** says that for any binary relation R on a non-empty set A such that for every $a \in A$ there exists a $b \in A$ such that aRb, for every $a \in A$ there exists a function $f : \omega \longrightarrow A$ satisfying $f(n)Rf(n+1)$ for any $n \in \omega$ and $f(0) = a$.

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and the implications cannot be reversed

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A subset $B \subseteq X$ is called a **Bernstein set** if $|B| = |X \setminus B| = \mathfrak{c}$ and neither B nor $X \setminus B$ contains a perfect subset.

Theorem 1 (F. Bernstein [1])

If an uncountable Polish space X can be well-ordered, then there exists a Bernstein set $B \subseteq X$, i.e. $\mathbf{WR} \rightarrow \mathbf{BS}$.

- a Bernstein set is a classical example of a non-measurable set

Theorem 2 (F. Bernstein [1])

A Bernstein set does not possess the Baire Property and is not Lebesgue measurable, i.e. $\mathbf{BP} \rightarrow \neg\mathbf{BS}$ and $\mathbf{LM} \rightarrow \neg\mathbf{BS}$.

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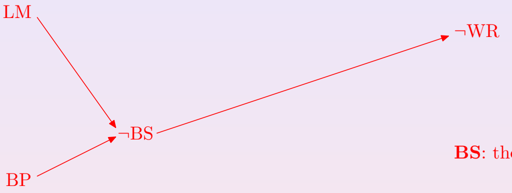

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Let $\langle X, +, 0\rangle$ be additive group. A set $V \subseteq X$ is called a **Vitali set** if there exists a countable dense subset D such that

- $(\forall x, y)\,((x, y \in V \wedge x \neq y) \to x - y \notin D)$,
- $(\forall x \in X)(\exists y \in V)\, x - y \in D$.

Note that, for every $x \in X$ there exists exactly one real $y \in V$ such that $x - y \in D$.
- the family $\{\{y \in X : x - y \in D\} : x \in X\}$ is a decomposition of the set X and we call it the **Vitali decomposition**
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If the real line can be well-ordered, then there exists a Vitali set, i.e. **WR** $\rightarrow$ **VS**.

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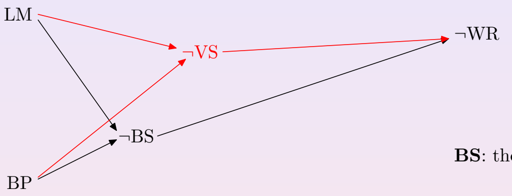

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of all finite subsets of ω is an ideal of algebra $\mathcal{P}(\omega)$.

- we can consider the quotient algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ and we denote by $\mathfrak{k}$ its cardinality

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Theorem 5

The inequalities $2^{\aleph_0} \leq \mathfrak{k}$ and $\mathfrak{k} \ll 2^{\aleph_0}$ hold true. Moreover, if the set $\mathcal{P}(\omega)$ can be well-ordered, then $\mathfrak{k} = 2^{\aleph_0}$, i.e. **In1** $\rightarrow \neg$**WR**.

Note the following: if A, B are sets such that $|A| \leq |B|, |B| \ll |A|$ then A can be well-ordered if and only if B can be well-ordered.

Corollary 6

A set of cardinality $\mathfrak{k}$ can be well-ordered if and only if the set of reals $\mathbb{R}$ can be well-ordered.

Corollary 7

If a set of cardinality $\mathfrak{k}$ cannot be linearly ordered, then $\aleph_1 < \aleph_1 + \mathfrak{c} < \aleph_1 + \mathfrak{k}$, i.e. $\neg$**LK** $\rightarrow$ **In2**.

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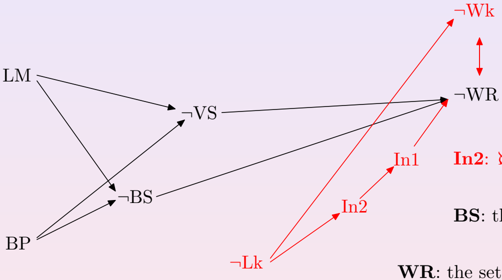

In1: $c < k \ll c$

In2: $\aleph_1 < \aleph_1 + c < \aleph_1 + k$

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Vitali set V on the Cantor space ${}^\omega 2$

- $(\forall x, y)\,((x, y \in V \wedge x \neq y) \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^\omega$,
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- the family

$$\{\{y \in {}^\omega 2 : \{n : x(n) \neq y(n)\} \in [\omega]^{<\omega}\} : x \in {}^\omega 2\}$$

is a Vitali decomposition of the Cantor space ${}^\omega 2$
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- Vitali decomposition: $\mathbb{T}/\mathbb{D} = \{\{y \in \mathbb{T} : x - y \in \mathbb{D}\} : x \in \mathbb{T}\}$
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- if there exists a selector for the Vitali decomposition, then a Vitali set is the set of cardinality $\mathfrak{k}$

Vitali set on the circle $\mathbb{T}$ for the set of all rational numbers $\mathbb{Q}$

$$\mathbb{T}/\mathbb{Q} \cong (\mathbb{T}/\mathbb{D})/(\mathbb{Q}/\mathbb{D})$$

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A set $A \subseteq \mathbb{T}$ is called a **tail-set** if the set $\{r \in \mathbb{T} : A + r = A\}$ contains a countable subset dense in $\mathbb{T}$.

Theorem 10 (J. Mycielski [1])

If AC_2 holds true, then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e. $\mathrm{LM} \to \neg\mathrm{AC}_2$ and $\mathrm{BP} \to \neg\mathrm{AC}_2$.

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If a set of cardinality $\mathfrak{c}$ is linearly ordered, then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e. $\mathrm{LM} \to \neg\mathrm{L}\mathfrak{c}$ and $\mathrm{BP} \to \neg\mathrm{L}\mathfrak{c}$.

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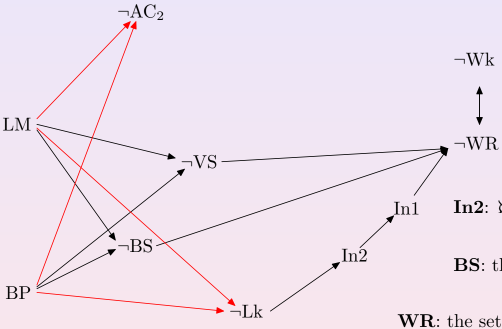

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A **free ultrafilter on** ω is a filter $\mathcal{J} \subseteq \mathcal{P}(\omega)$ not containing any finite set and for every $A \subseteq \omega$, either $A \in \mathcal{J}$ or $\omega \setminus A \in \mathcal{J}$.

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If the real line can be well-ordered, then there exists a free ultrafilter on ω, i.e. **WR** $\rightarrow$ **FU**.

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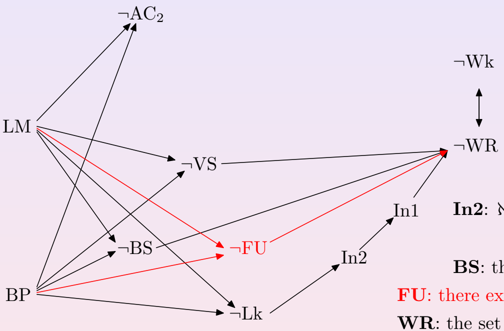

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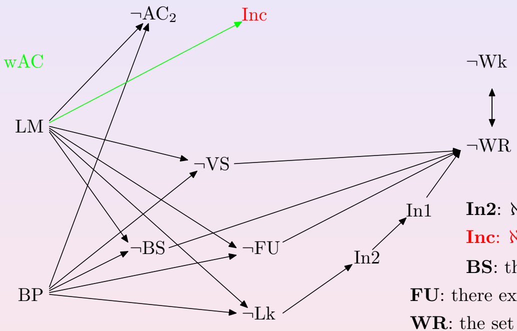
Regularity properties on the real line



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If **wCH** holds true, then the following are equivalent:

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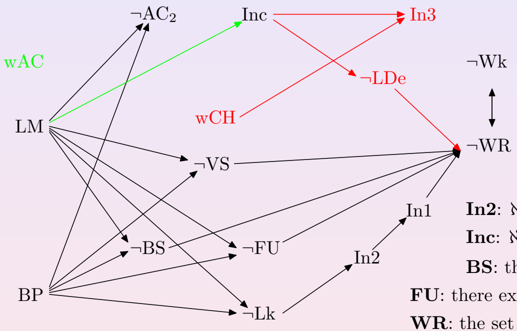

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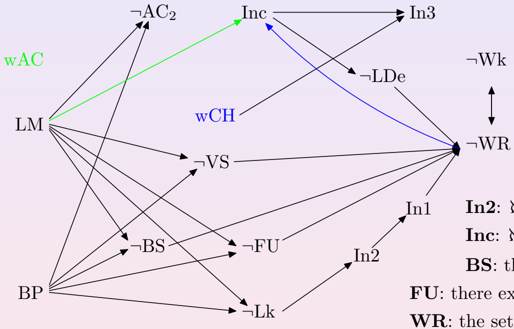

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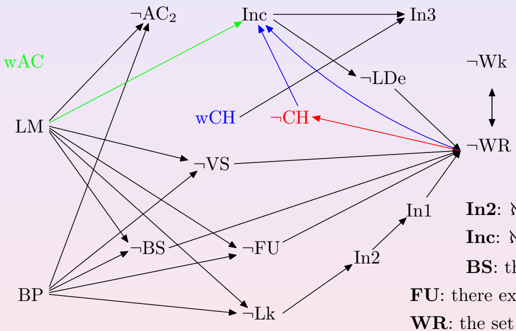

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Theorem 16

If every uncountable set of reals contains a perfect subset, then there is no set X such that $\aleph_0 < |X| < \mathfrak{c}$, i.e. **PSP** $\rightarrow$ **wCH**.

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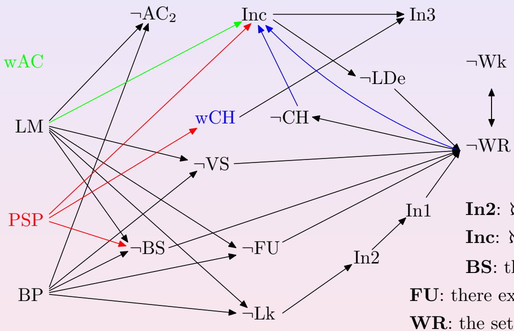

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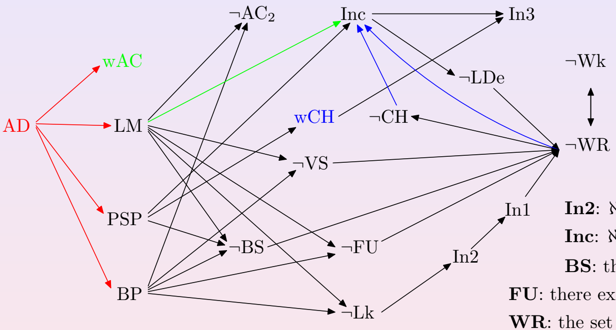

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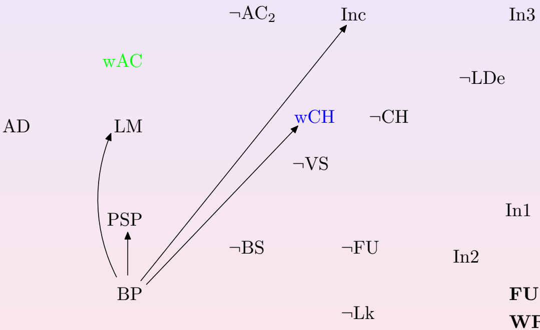

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If there is no selector for the Lebesgue decomposition and $\aleph_1$ is a regular cardinal, then $\aleph_1$ is an inaccessible cardinal in the constructible universe **L**.

Since $\aleph_1$ is not inaccessible in **L** in the Shelah's above mentioned model, we obtain

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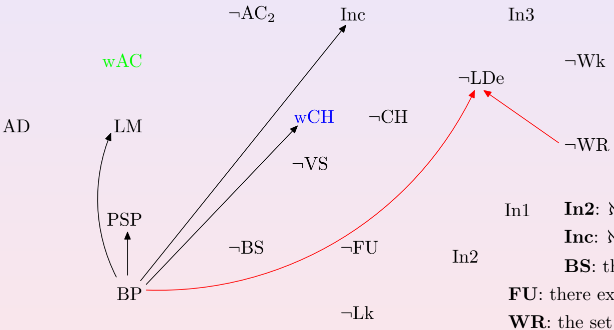

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